Lecture 9 — 22nd May 2025 — ①

IV. Clifford algebra, spin gp and spinor

IV.1 Clifford algebra

Definition: Let $(V, <,>)$ be a Euclidean space (of finite dim). Clifford alg. is a real algebra spanned by $1 \in \mathbb{R}$, $v \in V$ with the relation

$$v \cdot w + w \cdot v = -2\langle v, w\rangle \quad \forall\, v, w \in V$$

Explicit construction:

$$C(V) = \bigoplus_{k=0}^{+\infty} V^{\otimes k} \Big/ \text{two-sided ideal spanned by } v\otimes w + w \otimes v + 2\langle v, w\rangle$$

$$= T(V) \big/ I_{<,>}$$

Prop: Clifford algebra is the pair $(C(V), c)$ where $c : V \to C(V)$ with $c(v)^2 + |v|^2 = 0$ and $C(V)$ is the unique unital associative $\mathbb{R}$-algebra with the following universal property:

for any unital associative $\mathbb{R}$-algebra $\mathcal{A}$ and for any $\mathbb{R}$-linear map $\varphi : V \to \mathcal{A}$ s.t.

$$\varphi(v)\varphi(w) + \varphi(w)\varphi(v) = -2\langle v, w\rangle \quad \text{in } \mathcal{A}$$

then $\exists !\ \tilde{\varphi} : C(V) \to \mathcal{A}$ morphism of $\mathbb{R}$-algebra

$$\begin{tikzcd}
c(v) & C(V) \arrow[dr, dashed, "\exists ! \tilde{\varphi}"] & \\
v \arrow[u, mapsto] & V \arrow[u, "c"] \arrow[r, "\varphi"] & \mathcal{A}
\end{tikzcd}$$

Proof: – The existence by $T(V)/I_{\langle,\rangle}$
– uniqueness by the universal property. #

Proposition: We have a natural morphism of groups
$$O(V) \longrightarrow \mathrm{Aut}(C(V))$$
orthogonal gp automorphism group of $\mathbb{R}$-algebra

induced by $O(V) \curvearrowright V$.

proof: $O(V) \curvearrowright T(V) = \bigoplus_k V^{\otimes k}$

since $O(V) \curvearrowright V$ preserves the metric $\langle , \rangle$.

so that $A \in O(V)$, $v, w \in V$
$$A(v\otimes w + w \otimes v + 2\langle v, w\rangle)$$
$$= Av \otimes Aw + Aw \otimes Av + 2\langle Av, Aw\rangle \in I_{\langle,\rangle}$$
$\Rightarrow$ A preserves $I_{\langle,\rangle}$

$\Rightarrow$ $A \curvearrowright C(V) = T(V)/I_{\langle,\rangle}$ #

Def: $-Id_V \in O(V)$, Let $\tau \in \mathrm{Aut}(C(V))$ be the induced automorphism, so that $\tau^2 = Id$.

Proposition/Definition:
Let $C^{\pm}(V) \subset C(V)$ be the ± 1-eigenspace of τ.
Then $C(V) = C^{+}(V) \oplus C^{-}(V)$

and
$$C^+(V) = \left\langle c(v_1)\cdots c(v_k) : \begin{matrix} k \text{ even} \\ v_1, \cdots, v_k \in V \end{matrix} \right\rangle$$
$$= \operatorname{Im}\left(T^{\text{even}}(V) \to C(V) \right)$$
$$C^-(V) = \operatorname{Im}\left(T^{\text{odd}}(V) \to C(V) \right)$$

#

Proposition : $C(V) = C^+(V) \oplus C^-(V)$ is a superalgebra.

Pf : By previous proposition & Definition of $C^{\pm}(V)$

It is enough to verify
$$C^+(V)\,C^+(V),\quad C^-(V)\,C^-(V) \subset C^+(V)$$
$$C^-(V)\,C^+(V),\quad C^+(V)\,C^-(V) \subset C^-(V)$$

#

$V \cong \mathbb{R}^m$

Example : ① $V = \mathbb{R}$
$$C^+(\mathbb{R}) = \mathbb{R} \ni 1$$
$$C^-(\mathbb{R}) = V = \mathbb{R} \ni e \qquad e^2 = -1$$
$$\Rightarrow \quad C(\mathbb{R}) = \mathbb{C}$$
$$e \mapsto \sqrt{-1}$$

② $V = \mathbb{R}^2$ $\quad e_1, e_2$ ONB
$$\begin{cases} C^+(V) = \mathbb{R} \oplus \mathbb{R}\, e_1 e_2 \\ C^-(V) = \mathbb{R} e_1 \oplus \mathbb{R} e_2 \end{cases}$$
where $e_1^2 = e_2^2 = -1$
$$(e_1 e_2)^2 = e_1 e_2 e_1 e_2 = -e_1^2 e_2^2 = -1$$

$C(\mathbb{R}^2) \simeq \mathbb{H}$ quaternion number $\quad$ HW 3.3

$$C(V) = T(V) \Big/ \langle v\otimes w + w\otimes v + 2\langle v, w\rangle \rangle$$

Proposition: For V, W two Euclidean spaces, then the map

$$f: V\oplus W \longrightarrow C(V)\hat{\otimes} C(W)$$
$$(v, w) \longmapsto v\otimes 1 + 1\otimes w$$

extends to an isomorphism $f: C(V\oplus W) \cong C(V)\hat{\otimes} C(W)$

Proof: $$f(v,w)^2 = (\underset{-}{v}\hat{\otimes}\underset{+}{1} + \underset{+}{1}\hat{\otimes}\underset{-}{w})(v\hat{\otimes}1 + 1\hat{\otimes}w)$$
$$= v^2\hat{\otimes}1 + v\hat{\otimes}w - v\hat{\otimes}w + 1\hat{\otimes}w^2$$
$$= -\langle v, v\rangle - \langle w, w\rangle$$
$$= -\langle (v,w), (v,w)\rangle$$

$\Rightarrow \exists!\ f: C(V\oplus W) \longrightarrow C(V)\hat{\otimes} C(W)$ morphism

$i_1: V \hookrightarrow V\oplus W$

$\rightsquigarrow i_1: C(V) \rightarrow C(V\oplus W)$

$i_2: W \hookrightarrow V\oplus W$

$\rightsquigarrow i_2: C(W) \rightarrow C(V\oplus W)$

$(i_1\otimes i_2)\circ f: C(V\oplus W) \circlearrowleft$ s.t. it is $Id_{V\oplus W}$ on $V\oplus W$.

$\Rightarrow f$ is isomorphism of algebras #

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As a consequence

$$C(\mathbb{R}^m) = C(\mathbb{R}e_1)\hat{\otimes} C(\mathbb{R}e_2)\hat{\otimes}\cdots\hat{\otimes} C(\mathbb{R}e_m)$$

$$\Rightarrow \quad \dim_{\mathbb{R}} C(\mathbb{R}^m) = 2^m$$

Rk: We have an isomorphism (usual tensor product)

$$C(\mathbb{R}^{m+2})\otimes_{\mathbb{R}}\mathbb{C} \simeq (C(\mathbb{R}^m)\otimes_{\mathbb{R}}\mathbb{C})\otimes_{\mathbb{C}}(C(\mathbb{R}^2)\otimes_{\mathbb{R}}\mathbb{C})$$

$$\begin{cases} e_j \longmapsto e_j\otimes\sqrt{-1}\,e_{m+1}e_{m+2} & j=1,\cdots,m \\ e_{m+1} \longmapsto 1\otimes e_{m+1} \\ e_{m+2}\longmapsto 1\otimes e_{m+2} \end{cases}$$

#

Set $\quad C^k(V) = \operatorname{Im}\left(\bigoplus_{j\le k} V^{\otimes j}\to C(V)\right)\subset C(V)$

Then $\quad \mathbb{R} = C^0(V)\subset C^1(V)\subset\cdots\subset C^k(V)\subset\cdots$

defines a filtration of $C(V)$, and we have

$$C^i(V)\,C^j(V)\subset C^{i+j}(V).$$

The $\mathbb{Z}$-graded algebra associated with the above filtration of $C(V)$ is

$$Gr^\bullet(C(V)) = \bigoplus_k Gr^k(C(V))$$

with $\quad Gr^k(C(V)) := C^k(V)/C^{k-1}(V)$

and $\quad Gr^i\cdot Gr^j\subset Gr^{i+j}$

Proposition: We have identification of $\mathbb{Z}$-graded algebras

$$Gr^\bullet(C(V))\simeq \Lambda^\bullet V$$

Proof : $\mathrm{Gr}^0(C(V)) = C^0(V) = \mathbb{R} = \Lambda^0 V$

$V \hookrightarrow C(V)$ injective

$c(v)^2 = -|v|^2 = 0$ iff $v=0$

$\Rightarrow \quad C^1(V) = V \oplus \mathbb{R}$

$\mathrm{Gr}^1(C(V)) = C^1(V)/C^0(V) \simeq V = \Lambda^1 V$

Then $\mathrm{Gr}^\cdot(C(V))$ is generated by $\mathrm{Gr}^0(C(V)) = \mathbb{R}$

and $\mathrm{Gr}^1(C(V)) = V = \Lambda^1 V$

with the relation $c(v)c(w) + c(w)c(v) = -2\langle v, w\rangle$

$= 0$ in $\mathrm{Gr}^2(C(V))$

$= C^2(V)/C^1(V)$

$\Longleftrightarrow \quad v \wedge w + w \wedge v = 0$

By definition of $\Lambda^\cdot V$

$\Rightarrow$ isomorphism ! #

Def : $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$ real/complex $C(V)$-module

if 1) E is superspace

2) E is $C(V)$-module

3) $\begin{cases} C^+(V) E^\pm \subset E^\pm \\ C^-(V) E^\pm \subset E^\mp \end{cases}$

we call E a Clifford module

Rmk: Since $c(v)^2 = -|v|^2$ for $v \neq 0$

then $c(v): E^{\pm} \xrightarrow{\sim} E^{\mp}$

$\Rightarrow \dim E^+ = \dim E^-$ #

Proposition: For $v \in V$, define ↙ superalgebra

$$c(v) := v^* \wedge - \iota_v \in \mathrm{End}(\wedge^\cdot V^*)$$

where $v^* \in V^*$ is the metric dual of v

ι_v is the interior product, that is $\alpha \in V^*$

$$\iota_v \alpha = \alpha(v).$$

This makes $\wedge^\cdot V^*$ a $C(V)$-module.

proof: When acting on $\wedge^\cdot V^*$

$$
\begin{aligned}
& c(v)c(w) + c(w)c(v) \\
&= (v^*\wedge - \iota_v)(w^*\wedge - \iota_w) + (w^*\wedge - \iota_w)(v^*\wedge - \iota_v) \\
&= -\iota_v w^* - v^* \iota_w - \iota_w v^* \wedge - w^* \iota_v \\
&= -(\langle v, w\rangle + \langle w, v\rangle) = -2\langle v, w\rangle
\end{aligned}
$$

So we have

$$
\begin{tikzcd}
C(V) \arrow[dr, "\exists!\, c"] & \\
V \arrow[u, hook] \arrow[r, "c"] & \mathrm{End}(\wedge^\cdot(V^*))
\end{tikzcd}
$$
#

Def: $\sigma : C(V) \longrightarrow \wedge^\cdot V^*$

$$v \longmapsto c(v)\cdot 1 \qquad 1 \in \wedge^0 V^* = \mathbb{R}$$

called symbol map.

Prop: σ is an isomorphism of vector space

Proof: $e_1, \cdots, e_m$ ONB of $V = \mathbb{R}^m$

for $1 \le i_1 < \cdots < i_k \le m$

$$\sigma(c(e_{i_1}) \cdots c(e_{i_k})) = e^{i_1} \wedge \cdots \wedge e^{i_k} \in \Lambda^k V^*$$

#

Now we give the map σ^{-1}, called "quantization" map

Proposition/Definition:

$$\sigma^{-1}: \Lambda^\bullet V^* \longrightarrow C(V)$$

where if $\alpha \in \Lambda^p V^*$

$$\sigma^{-1}(\alpha) = \sum_{i_1 < \cdots < i_p} \alpha(e_{i_1}, \cdots, e_{i_p})\, c(e_{i_1}) \cdots c(e_{i_p})$$

Proof:

$$c(e_{i_1}) \cdots c(e_{i_p})\, 1 = (e^{i_1}\wedge - \iota_{e_{i_1}}) \cdots (e^{i_p}\wedge - \iota_{e_{i_p}})\, 1$$
$$= e^{i_1} \wedge \cdots \wedge e^{i_p}$$

$$\Rightarrow \quad \sigma(\sigma^{-1}(\alpha)) = \sum_{i_1 < \cdots < i_p} \alpha(e_{i_1}, \cdots, e_{i_p})\, e^{i_1} \wedge \cdots \wedge e^{i_p} = \underset{\Lambda^p V^*}{\alpha}$$

#

IV.2 Spin group

Recall: A Lie gp G is a gp G together with a smooth manifold structure on G s.t.

$$G \times G \longrightarrow G \qquad \text{and} \qquad G \longrightarrow G$$
$$(g, h) \longmapsto gh \qquad\qquad\quad g \longmapsto g^{-1}$$

both are smooth maps.

Example: ① $GL(r, \mathbb{R})$ or $GL(r, \mathbb{C})$

② $O(n) = \{ A \in GL(n, \mathbb{R}) : {}^{T}\!A = A^{-1} \}$

$SO(n) = \{ A \in O(n) : \det A = 1 \}$

$O(n)$ are the isometries of $(\mathbb{R}^n, \langle , \rangle)$.

③ $U(m) = \{ A \in GL(m, \mathbb{C}) : \underbrace{A^*}_{= {}^{T}(\bar{A})} = A^{-1} \}$

$SU(m) = \{ A \in U(m) : \det A = 1 \}$

$\mathfrak{g}$ = Lie algebra of G

= left-invariant vector field on $G \simeq T_1 G$

$[\ ,\ ]$ Lie bracket = Lie bracket of vector fields

$O(m)$ and $SO(m)$ has the same Lie algebra

HW 3.4

$$so(m) = \{ A \in \underbrace{M_m(\mathbb{R})}_{m \times m} : {}^{T}\!A = -A \}$$

A are anti-symmetric endomorphism of $\mathbb{R}^n$

Lie bracket $[A, B] = AB - BA$

as $m \times m$ matrices